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The Design of Coils for the Production of Strong Magnetic Fields

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VIII. *The Design of Coils for the Production of Strong Magnetic Fields.*

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(*Communicated by Sir* Ernest Rutherford, *P.R.S.*)

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1. P. Kapitza* has recently described a method of producing magnetic fields of the order of half a million to a million gauss. The principle of the method is to pass an enormous amount of power into a solenoidal coil for a fraction of a second, the current being switched off from the coil before it has time to heat up seriously. Powers of 40,000 kilowatts are available for dissipation in the coil, and by this means magnetic fields of the order of half a million gauss are attainable.

One of the most difficult problems of the work is to construct a satisfactory coil. When the current in the coil is 30,000 amperes and the magnetic field 500,000 gauss, the electrodynamic forces on the strip of the coil are about 1·5 tons per centimetre length. It is quite easy, therefore, to attain pressures of the order of 7000 kgms./sq. cm. in the body of the coil, forces which are far beyond the elastic limit of ordinary copper. It is necessary, therefore, to make a very careful study of the exact magnitude of the forces and resulting pressure distribution in the coil and to determine carefully the type of coil which will have the maximum strength. The second problem is to design the coil so as not to exceed the safe temperature of the insulation used and to produce at the same time the strongest magnetic field with the power available. This problem will be discussed first.

2. *The Design of Coils for Maximum Efficiency.*

(*a*) *The Magnetic Field produced by a Coil.*—It was first shown by Maxwell† that to obtain a coil of maximum efficiency we have to use coils of a section defined by the polar equation $r^2 = A \sin \theta$, where θ is the angle between the axis and the radius vector to the point on the boundary of the section. We have at the same time to increase the section of the coil as we proceed outwards in winding. Both these requirements are difficult to carry out when dealing with large powers, and we shall therefore assume in the first place that coils of rectangular section are used, these being simplest to wind.

The method of determining the coil of maximum efficiency which is applied here was

* 'Roy. Soc. Proc.,' A, vol. 115, p. 658 (1927).

† 'Electricity and Magnetism,' vol. 2, p. 331, 2nd edition.

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first suggested by FABRY.* The essential feature of the method is to plot the different constants of the coil against two parameters defining the shape.

It is well known (GRAY, 'Absolute Measurements,' 2nd edition, p. 216) that for this type of coil the relation between the field produced and the constants of the coil is

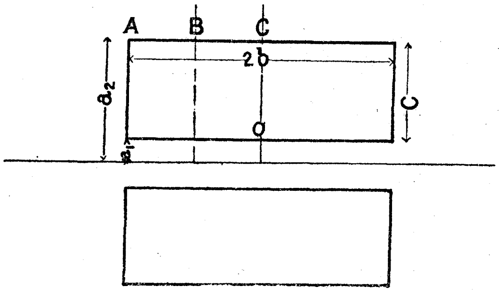

FIG. 1.

$$H = \frac{4\pi NI}{2c} \log \frac{a_2 + \sqrt{(b^2 + a_2^2)}}{a_1 + \sqrt{(b^2 + a_1^2)}}, \quad . \quad . \quad . \quad (1)$$

where a_1 and a_2 are the internal and external radii respectively, $2b$ is the axial length, N the number of turns, c the radial depth of winding.

The resistance of the coil is

$$R_2 = 2\pi N^2 (a_1 + a_2) \rho / 4bc\lambda, \quad . \quad . \quad . \quad . \quad (2)$$

where ρ is the specific resistance and λ the space factor.

Substituting for N in (1) we find

$$H = 2\sqrt{(2\pi)} \,.\, I \left\{\frac{bR_2\lambda}{c(a_1 + a_2)\rho}\right\}^{\frac{1}{2}} \log_e \frac{a_2 + \sqrt{(b^2 + a_2^2)}}{a_1 + \sqrt{(b^2 + a_1^2)}}.$$

If we now write $a_2/a_1 = \alpha$, $b/a_1 = \beta$, the shape of the coil is defined by α and β and the absolute dimensions solely by a_1 and we obtain

$$H = \sqrt{\left(\frac{R_2\lambda}{a_1\rho}\right)} IG_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where G_1 is a function of (α, β), varying only with the shape of the coil defined by

$$G_1 = 2\left(\frac{2\pi\beta}{\alpha^2 - 1}\right)^{\frac{1}{2}} \log_e \frac{\alpha + \sqrt{(\alpha^2 + \beta^2)}}{1 + \sqrt{(1 + \beta^2)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

To determine the relative efficiency of different types of coils we have therefore to find numerical values of the efficiency factor G_1 over a range of α and β from 0 to 20. The results are given in the G_1 curves of chart 1, curves of constant G_1 being plotted against (α, β). The maximum value of G_1 is 0·179.

We may note in passing that since the power W dissipated in the coil is i^2R_2, (3) may be written

$$H = \sqrt{\left(\frac{W\lambda}{a_1\rho}\right)} G_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

i.e., the maximum field attainable varies as the square root of the rate of power dissipation

* 'Eclairage Electrique' (Oct. 22, 1898).

in the coil and inversely as the square root of the internal radius. It is necessary, therefore, to make coils with a small internal radius.

(*b*) *The Source of Power and the Power Dissipation in the Coil.*—The source of power used in the experiments is an alternating current generator constructed with several modifications in design from the ordinary commercial alternator in order to obtain a very large ratio of transient short circuit current to steady short circuit current. The first half cycle of the short circuit current is then used to provide the current for the experiment, and as the current drops to zero the machine is switched off from the coil by an automatic switch.

If the E.M.F. of the machine on open circuit is given by $e = (E_e + E_s) \sin \omega t$, then if the machine is connected to a coil of self induction L_2 and resistance R_2 at the instant of zero voltage, the current is given by (*cf.* MILES WALKER, 'Specification and Design of Dynamo Electric Machinery,' p. 128)

$$i = \frac{E_s + E_e e^{-\lambda t}}{\sqrt{(R^2 + \omega^2 L^2)}} \sin(\omega t - \phi) + \frac{(E_s + E_e)}{\sqrt{(R^2 + \omega^2 L^2)}} \sin \phi e^{-Rt/L}, \quad . \quad . \quad . \quad (6)$$

where $R = R_1 + R_2$, $L = L_1 + L_2$ and $\phi = \text{arc tan } \omega L/R$, L_1 and R_1 being the effective self induction and resistance of the generator on short circuit. The E.M.F. is produced by two fields in the machine, E_s being the E.M.F. of constant amplitude produced by the resultant flux of the field coils and armature currents and E_e being a transient E.M.F. produced by the eddy current opposing the armature current in the poles of the machine. Since the damping factor of the eddy current has only a small effect during the first half cycle, we may write simply,

$$i = \frac{E}{\sqrt{(R^2 + \omega^2 L^2)}} \sin(\omega t - \phi) + \frac{E}{\sqrt{(R^2 + \omega^2 L^2)}} \sin \phi e^{-Rt/L}, \quad . \quad . \quad . \quad (7)$$

this current satisfying the circuit equation

$$L \frac{di}{dt} + Ri = E \sin \omega t$$

under the initial condition that $i = 0$ when $t = 0$.

To obtain the value of the maximum field produced by the current, we have to find the value of the current at its maximum. For circuits of small resistance the influence of the exponential would be small and we should have

$$i_{\text{max.}} = \frac{2E}{\sqrt{(R^2 + \omega^2 L^2)}},$$

the familiar doubling phenomenon. In general, putting $di/dt = 0$ for a maximum, we must have

$$\cos \phi e^{-Rt_m/L} = \cos(\omega t - \phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

to define t_m, the moment of maximum current, giving

$$i_{\text{max.}} = \frac{E}{\sqrt{(R^2 + \omega^2 L^2)}} \sin \omega t_{\text{max.}} \sec \phi = \frac{E}{\sqrt{(R^2 + \omega^2 L^2)}} f\left(\frac{X}{R}\right), \quad . \quad . \quad . \quad (9)$$

the value of the maximum current varying with X/R, the function f (X/R) having values from one to two as X/R goes from zero to infinity.

The magnetic field is, therefore, from (3), given by

$$H = \frac{E\sqrt{R_2}}{\sqrt{\{(R_1 + R_2)^2 + (X_1 + X_2)^2\}}} f\left(\frac{X_1 + X_2}{R_1 + R_2}\right)\left(\frac{\lambda}{a_1 \rho}\right)^{\frac{1}{2}} G_1(\alpha\beta). \quad . \quad . \quad . \quad (10)$$

In this equation X_2 and R_2 are not independent, X_2/R_2 being a function of (α, β), the shape of the coil which it is necessary to determine.

From data given in Scientific Paper No. 455 of the "Bureau of Standards" we may write for the self induction of a rectangular coil

$$L_2 = 2\pi^2 \frac{(1 + \alpha)^2}{4\beta} a_1 N^2 \kappa' \,.\, 10^{-9} \text{ henries,}$$

κ' being a tabulated function of $c/(a_1 + a_2)$ and b/c and therefore reduceable to a function of (α, β).

We have also from (2)

$$R_2 = \pi N^2 \frac{(1 + \alpha)}{(1 - \alpha)} \frac{\rho}{2\beta a_1 \lambda},$$

so that

$$\frac{L_2}{R_2} = \pi \frac{a_1^2 \lambda}{\rho} (\alpha^2 - 1) \kappa' \,.\, 10^{-9} = \frac{a_1^2 \lambda}{\rho} \,.\, 10^{-9} \phi_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

where ϕ_1 is a function of (α, β) only.

We therefore tabulate ϕ_1 as a function of (α, β) and plot curves of constant ϕ_1 on chart 1.

(*c*) *The Choice of Resistance for Maximum Field.*—The field is given by equation (10), in which R_2 is the only variable at our disposal. R_1 and X_1 are fixed constants of the machine, the relation between X_2 and R_2 is fixed by (A), a_1 the internal radius is fixed by experimental conditions, and α and β are to be supposed fixed by the heating of the coil, which will be discussed later.

For a maximum, therefore, we write $dH/dR_2 = 0$, and if Z be the total impedance we have

$$R_1^2 + X_1^2 = R_2^2 + X_2^2 + 2\left\{1 + \left(\frac{X_1 + X_2}{R_1 + R_2}\right)^2\right\} (X_1 R_2 - X_2 R_1) \frac{1}{f} \frac{\partial f}{\partial (X/R)}. \quad . \quad . \quad (11)$$

The quantity f'/f can be determined numerically, and it is therefore possible to find the value of R_2 to give the maximum field from (11). In actual practice the effect of the last term on the maximum is small, the maximum being very flat, and we get a value of R_2 giving very nearly the maximum field by neglecting the last term, which amounts to a neglect of the damping of the transient term and obtain, therefore, that the impedance of the machine has to equal the impedance of the coil.

So that

$$R_2 = Z_1/\{1 + \phi^2(\alpha\beta)\}^{\frac{1}{2}}, \quad \text{when} \quad X_2/R_2 = \phi(\alpha\beta) \quad \ldots \ldots \quad \text{(B)}$$

is the condition we use in practice for determining R_2.

As an example of the effect of the neglect of the last term in (11) we may take the case of a coil in which $\alpha = 6$, $\beta = 6$. Using (B) we find, taking values of R_1 and X_1, $2{\cdot}53 \,.\, 10^{-2}$ and $5{\cdot}08 \,.\, 10^{-2}$ respectively, the actual values for the machine,

$$R_2 = 3{\cdot}3 \,.\, 10^{-2}, \qquad X_2 = 4{\cdot}62 \,.\, 10^{-2}$$

and $(\sqrt{R_2/Z})\, f(X/R) = 1{\cdot}915$, this being proportional to the field from (10).

If we take two other neighbouring values of R_2, $3{\cdot}0 \,.\, 10^{-2}$ and $4{\cdot}0 \,.\, 10^{-2}$, we find values for $(\sqrt{R_2/Z})\, f(X/R)$ of $1{\cdot}90$ and $1{\cdot}875$ respectively, and for intermediate values of R_2 figures for $(\sqrt{R_2/Z})\, f(X/R)$ differing from $1{\cdot}915$ by amounts negligible in the practical coil. For a coil for which $\alpha = 16$ and $\beta = 8$, the resistance from (B) is $7{\cdot}4 \,.\, 10^{-3}$ ohms, but variations in this figure between the limits $6{\cdot}5 \,.\, 10^{-3}$ and $1{\cdot}2 \,.\, 10^{-2}$ produce variations in the field not greater than 1 per cent. at any point, the actual maximum being at a resistance of $9 \,.\, 10^{-3}$ ohms with an increase in the field of $0{\cdot}7$ per cent. over that corresponding to the resistance chosen from (B).

(*d*) *The Effect of the Deviation of the Wave Form from the Sinusoidal Form.*—In actual fact the wave of electromotive force of the machine on open circuit is not sinusoidal, the armature windings and field form being arranged to give a wave which is, neglecting high frequency tooth ripples, trapezoidal in form, being illustrated in fig. 2.

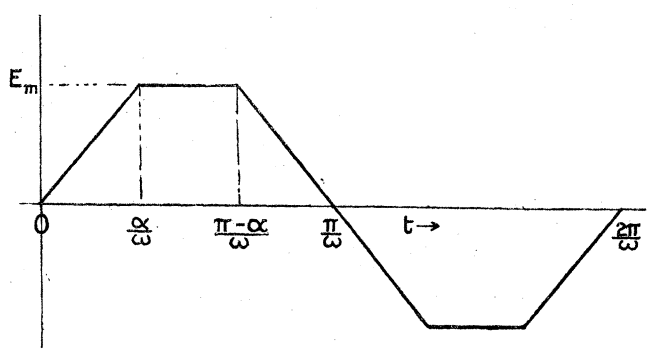

FIG. 2.

From $t = 0$ to $t = \alpha/\omega$ the E.M.F. rises linearly to a maximum value E_m, thence remains constant to $t = (\pi - \alpha)/\omega$, and after this drops linearly to $-E_m$ at $t = (\pi + \alpha)/\omega$.

The circuit equation may be written, putting $\omega t = \theta$,

$$L\omega\, di/d\theta + Ri = E\, \theta/\alpha,$$

subject to the initial condition that $i = 0$ for $\theta = 0$.

The solution is, therefore,

$$\left.\begin{aligned} i &= \frac{\mathrm{E}\theta}{\alpha\mathrm{R}} + \frac{\mathrm{EX}}{\alpha\mathrm{R}^2}\left(e^{-\mathrm{R}\theta/\mathrm{X}} - 1\right) & 0 \leq \theta \leq \alpha \\ i &= \frac{\mathrm{E}}{\mathrm{R}} + \frac{\mathrm{EX}}{\alpha\mathrm{R}^2}\left(1 - e^{\mathrm{R}\alpha/\mathrm{X}}\right) e^{-\mathrm{R}\theta/\mathrm{X}} & \alpha \leq \theta \leq (\pi - \alpha) \end{aligned}\right\}, \quad \ldots \ldots \quad (12)$$

with similar expressions for $\theta > (\pi - \alpha)$, and by differentiation we find that the maximum value of the current is given by

$$i_{\text{max.}} = \frac{\pi - \theta_{\text{max.}}}{\alpha} \cdot \frac{\mathrm{E}}{\mathrm{R}} = \frac{\mathrm{E}}{\mathrm{R}}\,\mathrm{F}\left(\frac{\mathrm{X}}{\mathrm{R}}\right), \quad \ldots \ldots \ldots \ldots \quad (\mathrm{C})$$

where $\theta_{\text{max.}}$ is defined by $\mathrm{R}\theta_{\text{max.}}/\mathrm{X} = e^{\mathrm{R}(\pi-\alpha)/\mathrm{X}} + e^{\mathrm{R}/\mathrm{X}} - 1$.

In fig. 3 numerical values of $(\pi - \theta_{\text{max.}})/\alpha$ or $\mathrm{F}(\mathrm{X}/\mathrm{R})$ are plotted as a function of X/R for $\alpha = \frac{1}{3}\pi$.

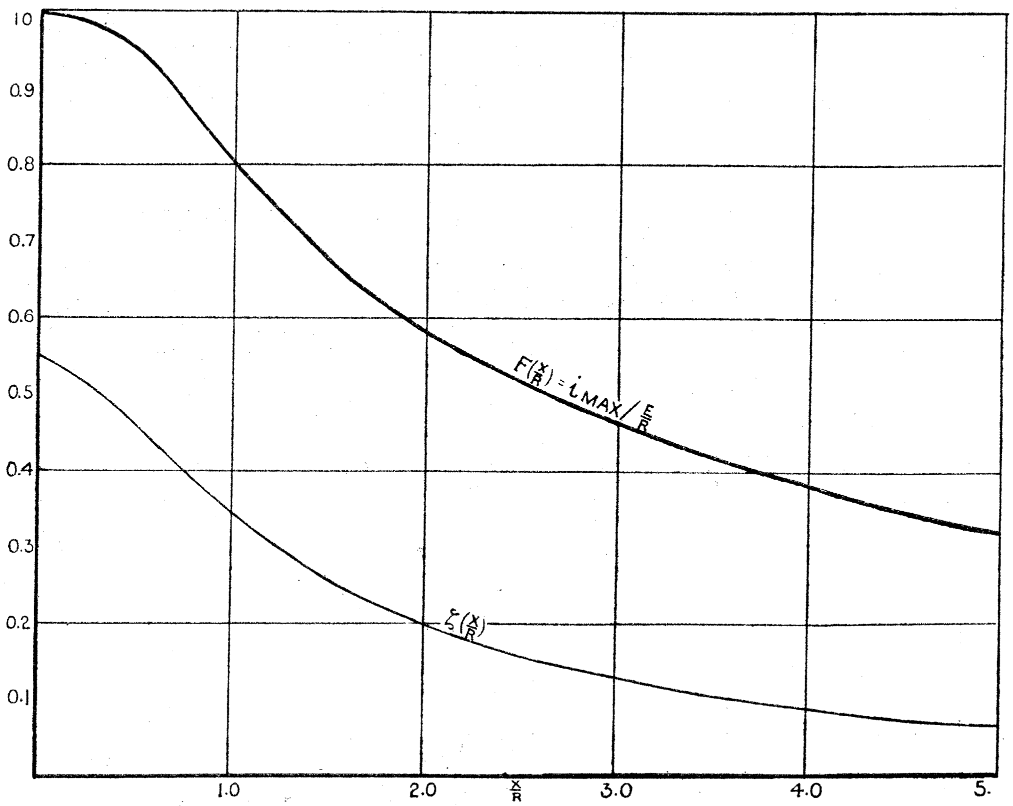

Fig. 3.

The field at the maximum is therefore, from (3), given by

$$\mathrm{H} = \frac{\mathrm{E}}{\mathrm{R}_1 + \mathrm{R}_2}\sqrt{\mathrm{R}_2}\,\mathrm{F}\left(\frac{\mathrm{X}}{\mathrm{R}}\right)\sqrt{\left(\frac{\lambda}{a_1\rho}\right)}\,\mathrm{G}_1. \quad \ldots \ldots \ldots \quad (\mathrm{D})$$

To find the value of R_2 giving the maximum field we should therefore proceed by differentiation of (D). Actually, however, since the fundamental harmonic term in the wave form is predominant, and since the maximum obtained by the method of section (*c*) is so very flat, we may use condition (B) in practice to choose R_2 without effecting the value of the maximum field attained by 1 per cent. As an example of this the resistance chosen by (B) for $\alpha = 6$, $\beta = 6$ is $3 \cdot 3 \,.\, 10^{-2}$ ohms. The relative values of field obtained from (D) for different values of R_2 near this figure are given in Table I.

TABLE I.

R_2	$3 \cdot 3 \,.\, 10^{-2}$	$3 \cdot 0 \,.\, 10^{-2}$	$2 \cdot 5 \,.\, 10^{-2}$	$2 \cdot 0 \,.\, 10^{-2}$	$4 \cdot 0 \,.\, 10^{-2}$
Field ...	1·000	1·000	1·000	0·985	0·990

We therefore use (D) to calculate the maximum of the field, using (B) to determine R_2.

(*e*) *The Heating of the Coil.*—If we were not limited by the heating of the coil, it is clear from (5) that there would be no limit to the field attainable, apart from limits set by the electrodynamical forces on the coil. Previous schemes for producing strong magnetic fields have therefore involved using liquid air as a cooling agent to limit the temperature rise produced. In the present experiments the temperature rise is limited by only allowing the current to flow in the coil for about $\frac{1}{100}$th of a second, but even in this short time, with powers of 40,000 kw. dissipated in the coil, temperature rises of 150° are attained, and although the insulation of the coil is mica and bakelite, it is inadvisable to proceed beyond this limit.

The temperature rise is given by

$$\delta T = \int_0^\tau i^2 R_2 \, dt \Big/ (V \times \text{density} \times \text{specific heat}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (E)$$

the current flowing in the coil from 0 to τ and the copper volume of the coil being denoted by V. During the short time of heating the dissipation of heat will be negligible.

The copper volume is given by

$$V = 2\pi a_1^3 (\alpha^2 - 1) \beta \,.\, \lambda = a_1^3 \lambda V_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (F)$$

where V_1 is a function of (α, β) plotted on chart 1, in the form of curves of constant V_1. To determine the temperature rise, we have now to know only the value of the integral

$$\int_0^\tau i^2 R_2 \, dt.$$

This integral cannot be calculated simply, for R_2 is a function of t determined by the power input to the coil, varying by 50 per cent. during the passage of the current. We do not, however, require to know the temperature rise accurately, and it will be sufficient to take a mean value of R_2. Taking the values of i given by (12), we find if θ_0 is the angle of zero current

$$\frac{\omega}{\pi}\int_0^{\tau} i^2 R_2\, dt = \frac{1}{\pi}\int_0^{\theta_0} i^2 R_2\, d\theta = \frac{E^2}{R^2} R_2 \zeta(X/R).$$

When $X/R > 1 \cdot 7$, $\theta_0 > \frac{4}{3}\pi$, and the function $\zeta(X/R)$ is given by

$$\zeta(X/R) = \frac{X}{R}\left\{\frac{1}{2} + \log_e \frac{X}{\alpha R}(1 - e^{y} - e^{2y} + e^{4y})\right\} + \frac{X^3}{2\alpha^2 R^3}(4 - 4e^{-2y} - 2e^{-3y} + 2e^{-4y}) - \frac{3X^2}{\alpha^2 R^2} - 2\alpha, \quad . \quad . \quad \text{(G)}$$

where $y = R\alpha/X$. For $X/R < 1 \cdot 7$ a more complex expression is obtained.

The function $\zeta(X/R)$ is plotted in fig. 3 and enables the power dissipated in the coil to be determined sufficiently approximately for the purposes of coil design. When a coil has actually been selected, its temperature rise may be estimated more accurately by the use of an integraph which is available, or by numerical integration, R_2 being first determined as a function of t and the resulting values used for the determination of $\int i^2 R_2\, dt$ by the integraph.

(*f*) *The Use of the Chart for the Design of Coils.*—The use of figures similar to chart 1 for coil design, was first suggested by FABRY (*loc. cit.*), the curves giving the time constant being added in our chart, since we are dealing with alternating current. We first select by approximate calculation from the power available in the experiment a copper volume which will give the temperature rise allowable. This corresponds to a particular volume curve V_1 on chart I. Since all points on this curve have the same volume, we take a point on the curve which intersects the G_1 curve of greatest value, since this will give the largest field for a given power dissipation. Thus, if we work on the curve $V_1 = 1000$, we take $\alpha = 6 \cdot 4$, $\beta = 3 \cdot 8$. We now determine $\phi(\alpha, \beta)$, the ratio of X_2/R_2 from the ϕ_1 curves of the chart and then choose R_2 from the relation (B). We now find the power dissipation in the coil from (G) and estimate the temperature rise from (E). If this is too large, we proceed to a larger volume curve and repeat the process; if too small, to a smaller volume curve. In this way, by a process of trial and error, the coil to give the maximum field with a given temperature rise is found. The actual value of the field to be expected is obtained from (D).

The value of the maximum field varies very much with the temperature rise allowed. An increase in allowable temperature means a smaller and consequently more efficient coil together with an increase in maximum current for a given applied voltage, since the time factor of the coil is reduced. The maximum field can also be appreciably increased by using a smaller internal radius, but this, of course, increases the experimental

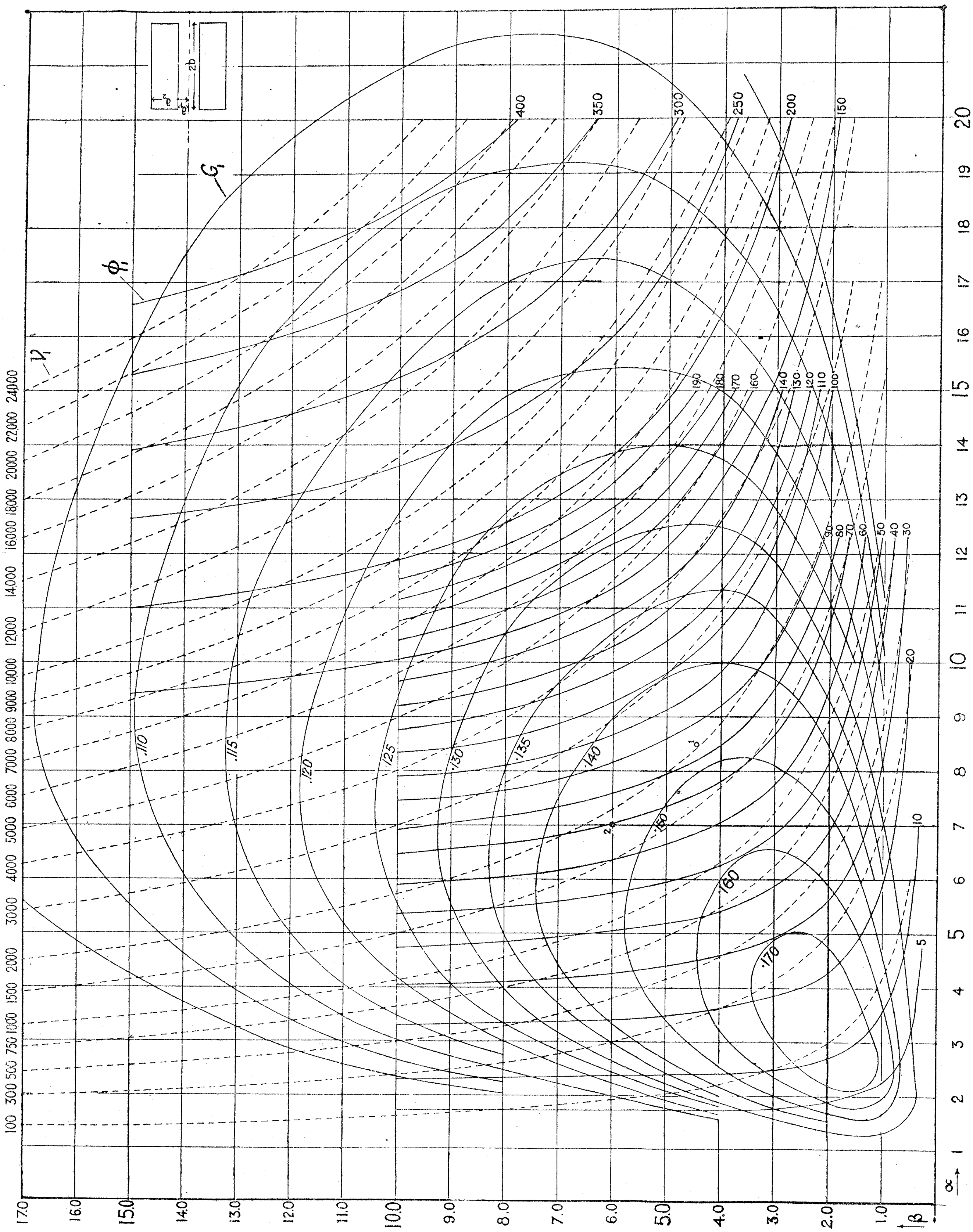

CHART I.—Volume of Copper $= a_1^3 V_1$. $L/R = a_1^2 \lambda/\rho \,.\, 10^{-9} \phi_1$. $H = i\,(R_2 \lambda/a_1\rho)^{\frac{1}{2}}\, G_1$.

difficulties. The effect of these two factors is shown in fig. 4, where the maximum field obtainable from the generator at full excitation is plotted as a function of allowable temperature rise and internal radius.

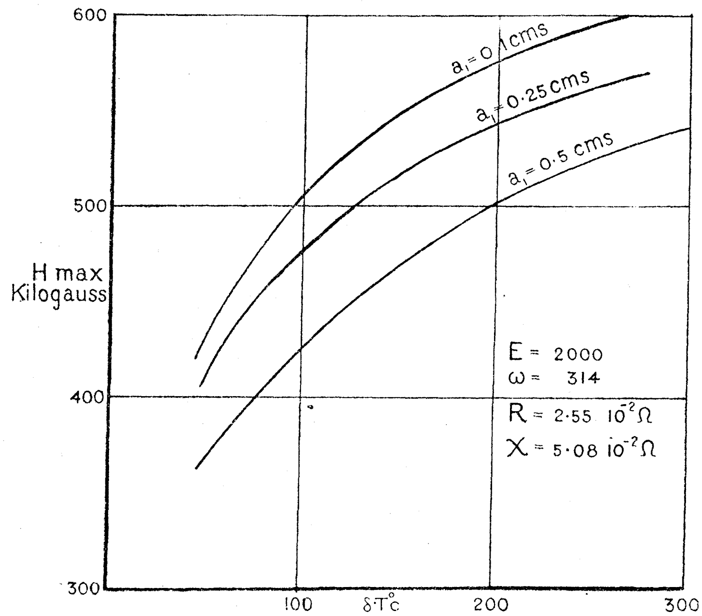

FIG. 4.

The effect of an increase in the power available can be investigated in a similar manner. If we did not have to consider the heating of the coil, then the maximum field would vary as the square root of the power available for dissipation in the coil. Actually, however, unless we use shorter times for the experiment or have resort to such devices as cooling the coil by liquid air, the coil will have to be made larger as we increase the power, and the resultant loss of efficiency so reduces the maximum field that an increase of power by a factor of 16 would be necessary to double the field. If we assume that an increase of power is obtained simply by a linear increase of the E.M.F. and impedance of the machine, then the value of the maximum field as a function of this E.M.F. is given in fig. 5. The result would not be much changed if the power increase were obtained by a decrease in impedance of the machine rather than an increase in voltage.

The effect of cooling the coil to liquid air temperature before an experiment would be first to increase the heat capacity for a given volume and so enable a smaller and more efficient coil to be used. Secondly, it would enable a coil with a greater number of turns and smaller wire section to be used since the mean resistivity would be lower, although for low temperatures and strong fields the increase of resistance of copper due to the field is appreciable. If the calculations are made it is found that by the use of a litre of liquid air the field attainable can be increased between thirty and forty per cent. The experimental difficulties attending its use would, however, be considerable due to the explosive character of the release of adsorbed gases from the coil.

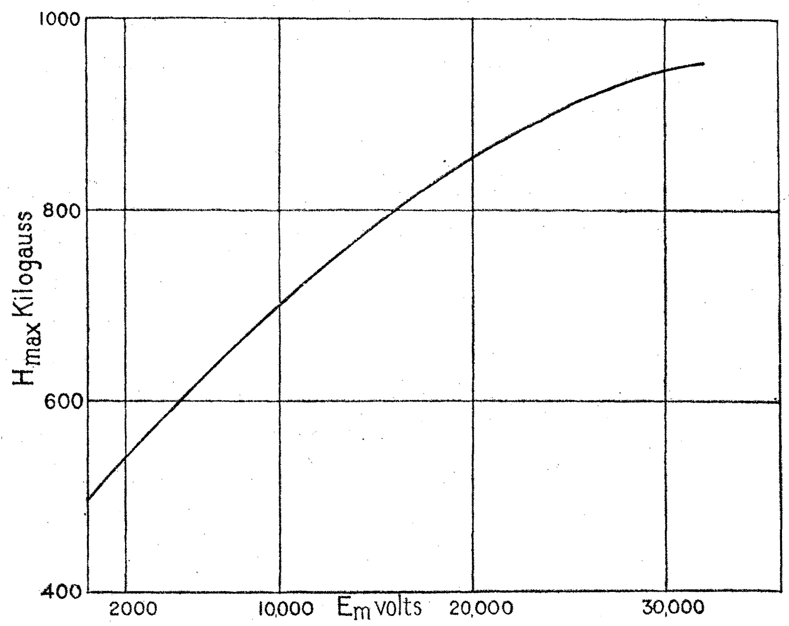

FIG. 5.

(3) *The Forces on the Coil.* (*Fig.* 6.)

Since we have at most points in the coil radial and axial magnetic fields, we have two sets of electrodynamical forces to consider—the axial compressive forces A and the

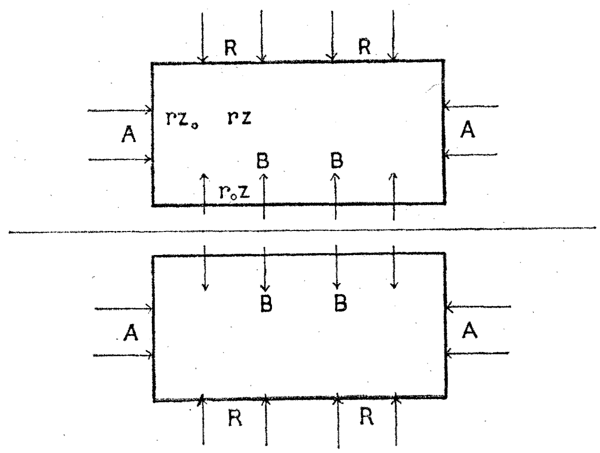

FIG. 6.

radial bursting forces B. In addition, we have the circumferential forces C due to the reinforcement.

A rough estimate of the magnitude of these forces shows that the pressures inside the copper of the coil may amount easily to 7000 kgms. per sq. cm.; far beyond the elastic limit of ordinary copper, the effect of these forces being shown by the bursting of the first coil, which was reinforced round its outside edge simply by a wooden frame. It is not very difficult to prevent the coil from bursting by supplying reinforcing forces R

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from a steel ribbon wound round the coil, but this in itself will not make a stable coil, since the copper will simply flow to the side under the forces B if not prevented by the forces A. If, on the other hand, we could reduce all the forces in the coil to a uniform hydrostatic pressure, then it is most likely that the coil would be stable. For this to occur we must have the principal stresses, due to A, B and C, equal at all points in the cross section. The problem, therefore, is to find a coil whose cross section has such a shape that the principal stresses in the material are as nearly equal as possible at all points.

The simplest way of deriving the forces was suggested by P. KAPITZA (*loc. cit.*). If M be the mutual inductance of the whole coil and a single turn in the section of the coil, then if i be the current density in the body of the coil and D the number of turns per unit sectional area of the coil, the radial and axial forces acting on an element of section dr, dz are

$$F_r = \frac{i^2}{2\pi r D}\frac{\partial M}{\partial r}\,dr\,dz \quad \dots \dots \dots \dots \quad (13)$$

$$F_z = \frac{i^2}{2\pi r D}\frac{\partial M}{\partial z}\,dr\,dz. \quad \dots \dots \dots \dots \quad (14)$$

If, then, P_r and P_z are the stresses in the radial and axial directions, and P_θ the circumferential stress, we have

$$\frac{\partial}{\partial r}(rP_r) - P_\theta = \frac{i^2}{2\pi D}\frac{\partial M}{\partial r},$$

$$\frac{\partial P_z}{\partial z} = \frac{i^2}{2\pi D r}\frac{\partial M}{\partial z} \quad \dots \dots \dots \dots \quad (15)$$

For $P_r = P_z = P_\theta$, the first equation of (15) becomes

$$\frac{\partial P_r}{\partial r} = \frac{i^2}{2\pi D}\frac{1}{r}\frac{\partial M}{\partial r};$$

so that for the two equations to hold together

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial M}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial M}{\partial z}\right)$$

or

$$\frac{1}{r^2}\frac{\partial M}{\partial z} = 0,$$

which is impossible.

It is therefore impossible to reduce the forces to a uniform hydrostatic pressure. If, however, we neglect the forces C due to the reinforcement, probably the most stable condition will be when we have $P_r = P_z$ at all points in the cross section. There will then be the least tendency for the copper to flow in the radial and axial directions, whilst in the circumferential direction it will be prevented by the symmetry of the forces.

Neglecting P_θ in (15) we may integrate directly so that

$$\left.\begin{aligned} P_r &= \frac{i^2}{2\pi r D}\{M(r, z) - M(r_0, z)\} \\ P_z &= \frac{i^2}{2\pi r D}\{M(r, z) - M(r, z_0)\} \end{aligned}\right\}, \quad \ldots \ldots \ldots \quad (16)$$

since the forces are zero at the inside cylindrical surface of the coil and on the end planes.

Thus for $\bar{P}_r = \bar{P}_z$ we must have $M(r_0, z) = M(r, z_0$ for all r and z, *i.e.*, the M function must have a constant value over the inside surface of the coil, and we have to find a coil of such a shape that this is satisfied.

The determination of the exact shape of such a coil would be very laborious, and in practice we can only wind coils whose section is composed of a number of steps, as in fig. 8, so that we have first to find a method of finding the mutual inductance of such a

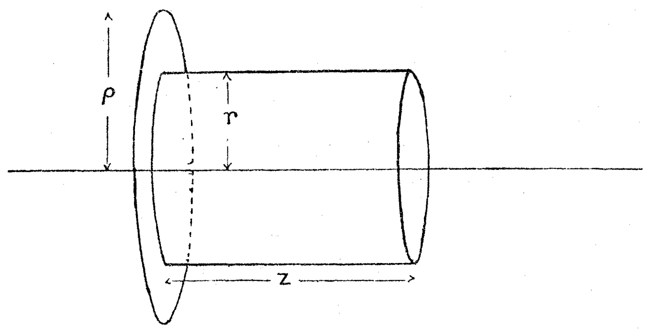

FIG. 7.

stepped coil and one turn on the inner surface, and then, by changing the dimensions of the steps, find a coil in which, at any rate, $\bar{P}_r - \bar{P}_z$ is small.

The method adopted is first to calculate and tabulate the function M for a solid coil and a single turn in its end plane (fig. 7) for different coil shapes and for all positions of the end turn. We may then obtain the function for any stepped coil by simple addition and subtraction of values obtained from the tables.

BUTTERWORTH* developed a method for calculating the coefficients of mutual induction of two solid coils, which is applicable, with modifications, to our problems.

We first calculate the coefficient of mutual induction for a semi-infinite solid coil of radius r and a coaxial circle of radius ρ distant z from the end plane of the solid coil. If Ω be the magnetic potential due to either the coil or ring, then it satisfies the equation

$$\frac{\partial^2\Omega}{\partial r^2} + \frac{1}{r}\frac{\partial\Omega}{\partial r} + \frac{\partial^2\Omega}{\partial z^2} = 0,$$

of which a solution is

$$\Omega = \int_0^\infty \phi(\lambda)\, e^{-\lambda z}\, J_0(\lambda r)\, d\lambda.$$

* 'Phil. Mag.,' vol. 29, p. 578 (1915).

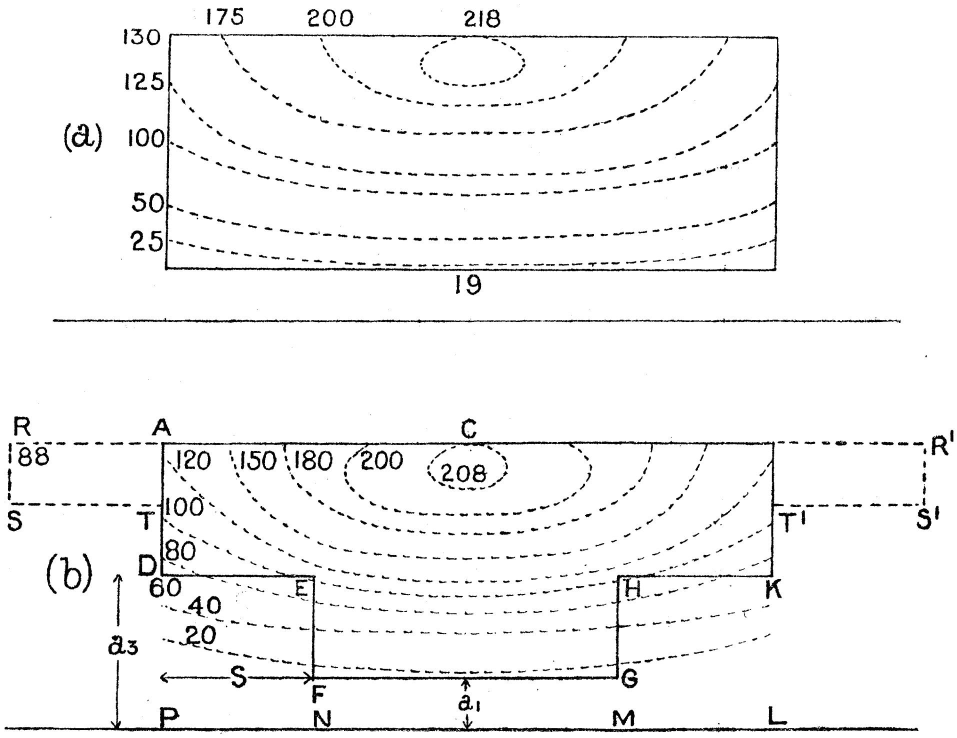

FIG. 8.

If we write ϕ for the flux of induction through a circle of radius r, then

$$\phi = -2\pi \int_0^r y \frac{\partial \Omega}{\partial z}\, dy,$$

and by expressing in terms of the magnetic potential along the axis, BUTTERWORTH showed that

$$\phi = -\pi r^2 \sum_{n=0}^{n=\infty} (-1)^n \left(\frac{r}{2}\right)^{2n} \frac{1}{n!\,(n+1)!} \frac{\partial^{2n+1} \Omega_0}{\partial z^{2n+1}}, \quad \ldots \ldots \quad (17)$$

where Ω_0 is the potential along the axis. The linkages of the field derived from Ω with a semi-infinite solid coil of radius r and D turns per square cm. are then

$$\begin{aligned} \mathrm{N} &= \int_0^r \int_2^\infty \phi \mathrm{D}\, dr\, dz \\ &= \pi \mathrm{D} r^3 \sum_0^\infty \frac{(-)^n}{2^{2n}} \frac{r^{2n}}{2n+3} \frac{1}{n!\,(n+1)!} \left(\frac{\partial^{2n+1}\Omega_0}{\partial z^{2n}}\right)_z^\infty \\ &= \pi \mathrm{D} r^3 \left[\frac{\Omega_0}{3} - \frac{r^2}{40} \frac{\partial^2 \Omega_0}{\partial z^2} + \frac{r^4}{1344} - \frac{\partial^4 \Omega_0}{\partial z^4} - \frac{r^6}{82{,}944} \frac{\partial^6 \Omega_0}{\partial z^6} \ldots\right]. \quad \ldots \quad (18) \end{aligned}$$

Since we are considering the mutual inductance of a solid coil and circle, we take for Ω_0 the magnetic potential of a single turn

$$\Omega_0 = 2\pi \left\{1 - \frac{z}{\sqrt{(\rho^2 + z^2)}}\right\}. \quad \ldots \ldots \quad (19)$$

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Writing $\xi = \sqrt{(1 + z^2/\rho^2)}$ we have now for z small

$$\frac{1}{2\pi}\frac{\partial^2\Omega_0}{\partial z^2} \doteq \frac{1}{\rho^3}\frac{3z}{\xi^5}, \text{ etc.,}$$

so that

$$\frac{N}{2\pi^2 r^3 D} = \frac{1}{3} - \frac{z}{3\rho\xi} - \frac{r^2}{40\rho^2}\cdot\frac{3z}{\rho\xi^5} - \frac{r^4 z}{1344\,\rho^5}\left(\frac{105}{\xi^9} - \frac{60}{\xi^7}\right)$$

$$- \frac{r^6 z}{82{,}944\,\rho^7}\left(\frac{10{,}395}{\xi^{13}} - \frac{11{,}340}{\xi^{11}} + \frac{2520}{\xi^9}\right)$$

$$- \frac{63 r^8 z}{180{,}224\,\rho^9}\left(\frac{715}{\xi^{17}} - \frac{1144}{\xi^{15}} + \frac{528}{\xi^{13}} - \frac{640}{\xi^{11}}\right)$$

$$- 2\cdot71\,.\,10^{-5}\frac{r^{10}z}{\rho^{11}}\left(\frac{20{,}995}{\xi^{21}} - \frac{44{,}200}{\xi^{19}} + \frac{31{,}200}{\xi^{17}} - \frac{8330}{\xi^{15}} + \frac{640}{\xi^{15}}\right). \quad (20)$$

This expansion is valid for $z < 2$ and $r/\rho < 1$.

For large values of z we expand Ω_0 in powers of $1/z$ and differentiate term by term and obtain

$$\frac{N}{2\pi^2 r^3 D} = \frac{1}{6}\frac{\rho^2}{z^2} - \frac{1}{8}\frac{\rho^4}{z^4}\left(1 + \frac{3}{5}\frac{r^2}{\rho^2}\right) + \frac{5}{48}\frac{\rho^6}{z^6}\left(1 + \frac{9}{5}\frac{r^2}{\rho^2} + \frac{3}{7}\frac{r^4}{\rho^4}\right)$$

$$- \frac{35}{384}\frac{\rho^8}{z^8}\left(1 + \frac{18}{5}\frac{r^2}{\rho^2} + \frac{18}{7}\frac{r^4}{\rho^4} + \frac{1}{3}\frac{r^6}{\rho^6}\right)$$

$$+ \frac{21}{256}\frac{\rho^{10}}{z^{10}}\left(1 + 6\frac{r^2}{\rho^2} + \frac{60}{7}\frac{r^4}{\rho^4} + \frac{10}{3}\frac{r^6}{\rho^6} + \frac{3}{11}\frac{r^8}{\rho^8}\right) + \ldots \quad (21)$$

This expansion is valid for all values of z greater than 2 and $r/\rho \leq 1$. For $r/\rho \geq 1$ we require different expansions.

We use (17) for the flux through the circle of radius ρ, giving

$$\phi = -\pi\rho^2 \sum_{n=0}^{n=\infty} (-1)^n \left(\frac{\rho}{2}\right)^{2n} \frac{1}{n!\,(n+1)!}\frac{\partial^{2n+1}\Omega_0}{\partial z^{2n+1}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (22)$$

and use for Ω_0 the magnetic potential due to the semi-infinite coil the end plane of which is at a distance z from the circle.

This is clearly given by

$$\Omega_0 = 2\pi\int_z^\infty\int_0^r\left(1 - \frac{z}{\sqrt{(y^2+z^2)}}\right)D\,.\,dz\,dy, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (23)$$

$$= \pi r^2 D\left\{\frac{z^2}{r^2}\log\frac{1+\zeta}{z/r} + \zeta - \frac{2z}{r}\right\}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (24)$$

if we take $\zeta = \sqrt{(1 + z^2/r^2)}$.

For large z we expand the integrand in (23) in inverse powers of z and integrate term by term and obtain

$$\frac{1}{2\pi D}\frac{\partial\Omega_0}{\partial z} = \frac{1}{6}\frac{r^3}{z^2} - \frac{3}{40}\frac{r^5}{z^4} + \frac{5}{112}\frac{r^7}{z^6}, \text{ etc.},$$

so that

$$\frac{N}{2\pi^2\rho^2 r D} = \frac{1}{6}\frac{r^3}{z^2} - \frac{3}{40}\frac{r^4}{z^4}\left(1 + \frac{3}{5}\frac{\rho^2}{r^2}\right) + \frac{5}{112}\frac{r^6}{z^6}\left(1 + \frac{21}{5}\frac{\rho^2}{r^2} + \frac{7}{3}\frac{\rho^4}{r^4}\right)$$
$$- \frac{35}{112}\frac{r^8}{z^8}\left(1 + \frac{54}{7}\frac{\rho^2}{r^2} + \frac{54}{5}\frac{\rho^4}{r^4} + 3\frac{\rho^6}{r^6}\right)$$
$$+ \frac{63}{2816}\frac{r^{10}}{z^{10}}\left(1 + \frac{110}{9}\frac{\rho^2}{r^2} + \frac{220}{7}\frac{\rho^4}{r^4} + 44\frac{\rho^6}{r^6} + \frac{22}{3}\frac{\rho^8}{r^8}\right). \quad \quad (25)$$

This expansion is valid for $z > 3$, $\rho/r \leq 1$.

For $\rho \leq z \leq 3$ we obtain $d^{2n+1}\Omega_0/dz^{2n+1}$ directly from (24).

We can write

$$\Omega_0 = \Omega_1 + \Omega_2,$$

where

$$\Omega_1 = \pi D\,[z^2 \log r\,(1+\zeta) + r^2\zeta - 2zr] \quad \text{and} \quad \Omega_2 = -\pi D z^2 \log z.$$

Then

$$-\frac{1}{2\pi D}\frac{\partial\Omega_2}{\partial z} = z\log z + \tfrac{1}{2}z,$$

$$-\frac{1}{2\pi D}\frac{\partial^{2n+1}\Omega_2}{\partial z^{2n+1}} = \frac{2n-2\,!}{z^{2n-1}} \qquad n > 2,$$

$$-\frac{1}{2\pi r D}\frac{\partial\Omega_1}{\partial z} = 1 - \frac{z}{r}\{\log r\,(1+\zeta) + \tfrac{1}{2}\}, \text{ etc.},$$

so that

$$\frac{N}{2\pi^2\rho^2 r D} = 1 - \frac{z}{r}\log\frac{1+\zeta}{z/r} - \frac{1}{8}\frac{\rho^2}{zr\zeta^3} - \frac{1}{192}\frac{z\rho^4}{r^5}\left\{\frac{1}{\zeta^3} + \frac{3}{\zeta^5} - \frac{15}{\zeta^7} - \frac{1}{\zeta\,.\,(1+\zeta)^2}\right\}$$
$$+ \frac{1}{3072}\frac{z\rho^6}{r^7}\left\{\frac{315}{\zeta^{11}} - \frac{245}{\zeta^9} - \frac{15}{\zeta^7} + \frac{3}{\zeta^5} - \frac{1}{\zeta^3} + \frac{1}{\zeta\,.\,(1+\zeta)^2} + \frac{2}{\zeta\,.\,(1+\zeta)^3}\right\}$$
$$+ 4{\cdot}06\,.\,10^{-6}\frac{z\rho^8}{r^9}\left\{\frac{45045}{\zeta^{15}} - \frac{58905}{\zeta^{13}} + \frac{18585}{\zeta^{11}} - \frac{525}{\zeta^9} - \frac{75}{\zeta^7}\right.$$
$$\left. + \frac{15}{\zeta^5} - \frac{15}{\zeta^3} + \frac{15}{\zeta\,.\,(1+\zeta)^2} + \frac{30}{\zeta\,.\,(1+\zeta)^3} + \frac{30}{\zeta\,.\,(1+\zeta)^4}\right\}$$
$$+ \frac{1}{96}\frac{\rho^4}{rz^3}\left\{1 - \frac{1}{4}\frac{\rho^2}{z^2} + 0{\cdot}094\frac{\rho^4}{z^4} - 0{\cdot}044\frac{\rho^6}{z^6} + 0{\cdot}023\frac{\rho^8}{z^8}\right.$$
$$\left. + 0{\cdot}008\frac{\rho^{12}}{z^{12}} - 0{\cdot}013\frac{\rho^{10}}{z^{10}} + 0{\cdot}008\frac{\rho^{12}}{z^{12}}\ldots\right\}. \quad \quad (26)$$

This series is valid for $\rho \leq z \leq 3$, $\rho/r < 1$. It converges rather slowly for $\rho/r = 1$ and z small.

For $z < \rho$, the last bracketed term in (26) ceases to converge. This term is derived from the successive derivatives of Ω_2, and we must therefore employ another method of calculating the linkages due to Ω_2.

BUTTERWORTH's method is applied to find these linkages.

The linkages due to an axial distribution of poles extending from $z = 0$ to $z = c$ with density $-\pi z^2 D$ is found. The potential due to these poles at $-z$ is

$$\begin{aligned}\Omega_3 &= D\int_0^c - \frac{\pi x^2}{(z+x)}\,dx \\ &= D\{-\pi z^2 \log z + \pi z^2 \log (c+z) - \pi cz + \tfrac{1}{2}\pi c^2\} \\ &= \Omega_2 + \Omega_4.\end{aligned}$$

If, now, we denote the linkages due to Ω_2, Ω_3, etc., by N_2, N_3, we have

$$N_3 = N_2 + N_4. \qquad (27)$$

N_4 is calculated by applying (22), and if c is very large we find

$$N_4 = \pi\rho^2 D\{\pi c - 2\pi z \log (c + z)\}. \qquad (28)$$

If, then, we can calculate N_3 by a direct integration, N_2 can be found. Since the flux through a circle of radius ρ due to a magnetic pole on the axis at distance y is $2\pi\,[1 - y/(\rho^2 + y^2)^{\frac{1}{2}}]$, we have

$$\begin{aligned}\frac{N_3}{D} &= 2\pi^2 \int_z^{c+z} (x-z)^2 \left\{1 - \frac{x}{\sqrt{(\rho^2 + x^2)}}\right\} dx \\ &= -2\pi^2\rho^2 z \log \frac{2(c+z)}{\rho} + 2\pi^2\rho^2 z + \pi^2\rho^2 c \\ &\quad - 2\pi^2 [\tfrac{1}{3}z^3 + (\tfrac{2}{3}\rho^2 - \tfrac{1}{3}z^2)\sqrt{(\rho^2+z^2)} - z\rho^2 \log \{(z + \sqrt{\rho^2+z^2})/\rho\}]. \qquad (29)\end{aligned}$$

Thus

$$\begin{aligned}N_2 = N_3 - N_4 &= -2\pi^2 D\rho^2 z \log (z/\rho) + 2\pi^2 D\rho^2 z \\ &\quad - 2\pi^2 D [\tfrac{1}{3}z^3 + (\tfrac{2}{3}\rho^2 - \tfrac{1}{3}z^2)\sqrt{(\rho^2+z^2)} - z\rho^2 \log \{(z + \sqrt{\rho^2+z^2})/\rho\}]. \qquad (30)\end{aligned}$$

We therefore have in place of (26)

$$\begin{aligned}\frac{N}{2\pi^2\rho^2 r D} &= 1 + \frac{z}{r}\left\{\tfrac{1}{2} - \log\frac{2(1+\zeta)}{\rho/r}\right\} + \tfrac{1}{8}\frac{z\rho^2}{r^3}\left(\frac{1}{\zeta\,.\,1+\zeta} + \frac{1}{\zeta^3}\right) \\ &\quad - \frac{1}{192}\frac{z\rho^4}{r^5}\left\{\frac{1}{\zeta^3} + \frac{3}{\zeta^5} - \frac{15}{\zeta^7} - \frac{1}{\zeta\,.\,(1+\zeta)^2}\right\} + \frac{1}{3072}\frac{z\rho^6}{r^7}A + 4\cdot 06\,.\,10^{-6}\frac{z\rho^8}{r^9}B \\ &\quad - \frac{\rho}{r}\left\{\frac{z^3}{3\rho^3} + \left(\frac{2}{3} - \frac{z^2}{3\rho^2}\right)\sqrt{\left(1 + \frac{z^2}{\rho^2}\right)} - \frac{z}{\rho}\log\frac{z + \sqrt{(\rho^2+z^2)}}{\rho}\right\}, \qquad (31)\end{aligned}$$

the terms A and B being the same as in (26).

This series is valid for $z < \rho < 1$.

From the five series given by (20), (21), (25), (26), (31) numerical values of N to cover the range likely to be required were calculated.

The figures obtained refer to the mutual inductance between a semi-infinite solid coil and a circle at distance z. If from this we subtract the mutual inductance of a system having the same radii but for which $z = 0$, we are left with the mutual induction of a solid coil of length z and a circle in the plane of the end face. This has been carried out and the final results are given in Tables II and III. For $z/r = 0 \cdot 5$ and $0 \cdot 25$ and values of $\rho/r = 1$ the convergence of the series used is slow and results cannot be obtained with an accuracy greater than one or two parts in a thousand. The figures given in the table for $\rho/r = 1$ obtained by the two series differ by this order. From these tables,

TABLE II.—Mutual Inductance of Solid Coil and Circle in End Plane.

$M/2\pi^2 r^3 D$.

ρ/r	z/r ∞	10	6	4	3	2	1	0·5	0·25
1·0	0·33333	0·33169	0·32886	0·32362	0·31689	0·30063	0·24825	0·1737	0·1065
0·9	0·32400	0·32266	0·32036	0·316071	0·31051	0·29685	0·25095	0·1821	0·1174
0·8	0·29867	0·29762	0·29579	0·29236	0·28788	0·27673	0·23781	0·1762	0·1130
0·7	0·21633	0·26052	0·25911	0·25647	0·25299	0·24418	0·21241	0·1595	0·1030
0·6	0·21600	0·21540	0·21437	0·21241	0·20980	0·20317	0·17845	0·13549	0·08804
0·5	0·16666	0·16625	0·16552	0·16416	0·16233	0·15761	0·13957	0·10706	0·07007
0·4	0·11733	0·11707	0·11660	0·11572	0·11454	0·11146	0·09943	0·07706	0·05098
0·3	0·07200	0·07186	0·07159	0·07109	0·07042	0·06866	0·06167	0·04835	0·03245
0·2	0·034666	0·03460	0·03448	0·03426	0·03396	0·03317	0·02998	0·02381	0·016306
0·1	0·009333	0·009316	0·009288	0·009230	0·009157	0·008958	0·008151	0·006569	0·004615

TABLE III.—Mutual Inductance of Solid Coil and Circle in End Plane.

$M/2\pi^2 r^3 D$.

ρ/r	z/ρ ∞	10	4	3	2	1	0·5	0·25
10	0·33333	0·33168	0·32338	0·31623	0·29814	0·23583	0·14928	0·08101
9	0·33333	0·33168	0·32338	0·31623	0·29814	0·23586	0·14933	0·08105
8	0·33333	0·33168	0·32338	0·31623	0·29814	0·23590	0·14940	0·08110
7	0·33333	0·33168	0·32338	0·31624	0·29815	0·23597	0·14950	0·08118
6	0·33333	0·33168	0·32338	0·31625	0·29818	0·23606	0·14966	0·08130
5·5	0·33333	0·33168	0·32339	0·31625	0·29820	0·23614	0·14978	0·08138
5·0	0·33333	0·33168	0·32339	0·31626	0·29822	0·23623	0·14994	0·08149
4·5	0·33333	0·33168	0·32339	0·31626	0·29824	0·23635	0·15014	0·08165
4·0	0·33333	0·33168	0·32339	0·31627	0·29829	0·23654	0·15043	0·08188
3·5	0·33333	0·33168	0·32340	0·31628	0·29834	0·23677	0·15084	0·08220
3·0	0·33333	0·33168	0·32340	0·31630	0·29843	0·23717	0·15150	0·08271
2·5	0·33333	0·33168	0·32342	0·31634	0·29856	0·23781	0·15260	0·08358
2·0	0·33333	0·33169	0·32344	0·31640	0·29880	0·23898	0·15468	0·08528
1·5	0·33333	0·33169	0·32348	0·31653	0·29930	0·24149	0·15943	0·08940
1·0	0·33333	0·33169	0·32362	0·31690	0·30062	0·24821	0·1740	0·1067

ρ = circle diameter. r = coil diameter. z = coil axial length.

charts II, III, IV and V were plotted. Charts II and III give the mutual inductance of a solid coil and a circle in the plane of the end, the circle radius being less than the coil

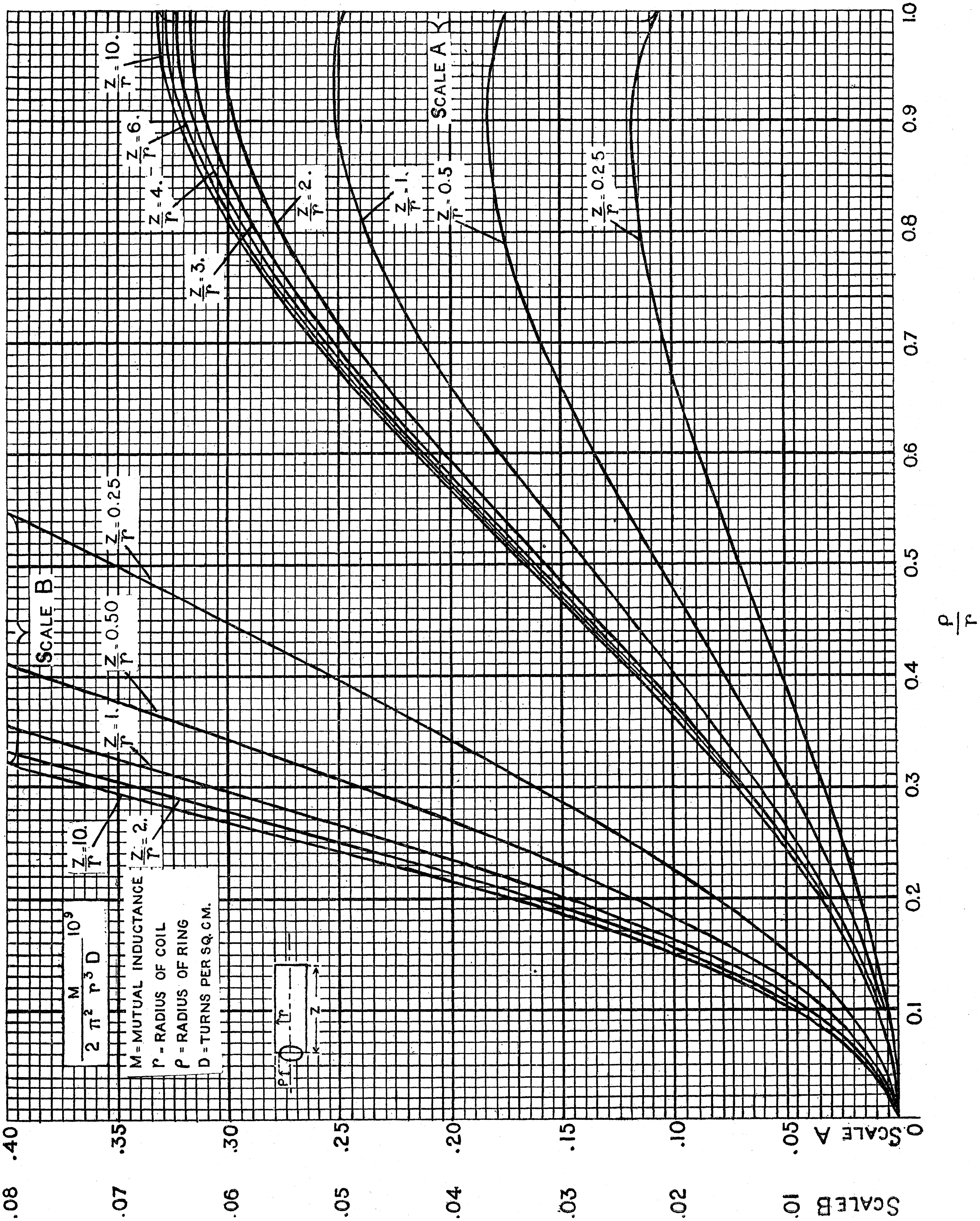
M = MUTUAL INDUCTANCE
r = RADIUS OF COIL
ρ = RADIUS OF RING
D = TURNS PER SQ. C.M.
SCALE A
SCALE B
ρ/r

Chart II.

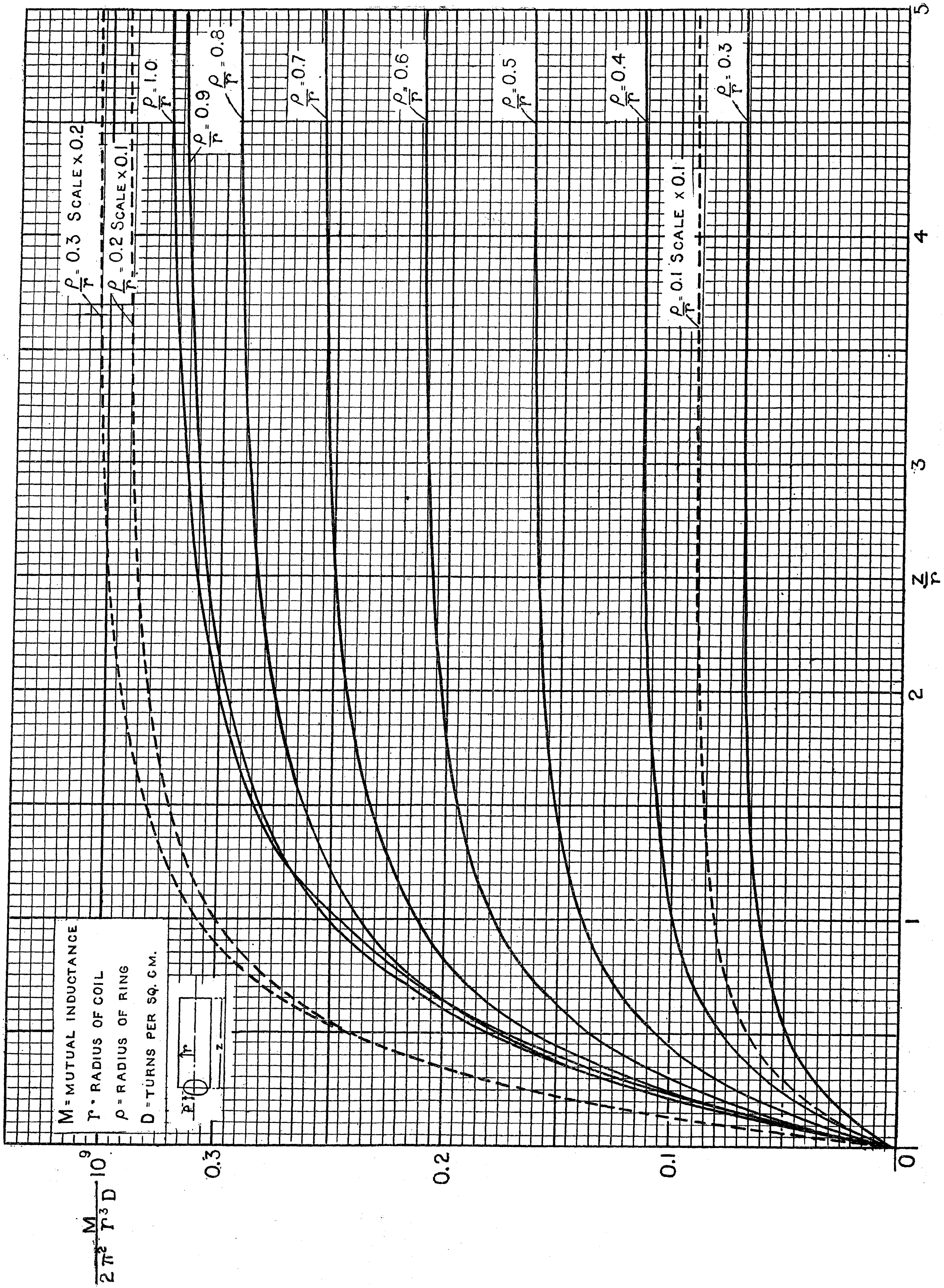

CHART III.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

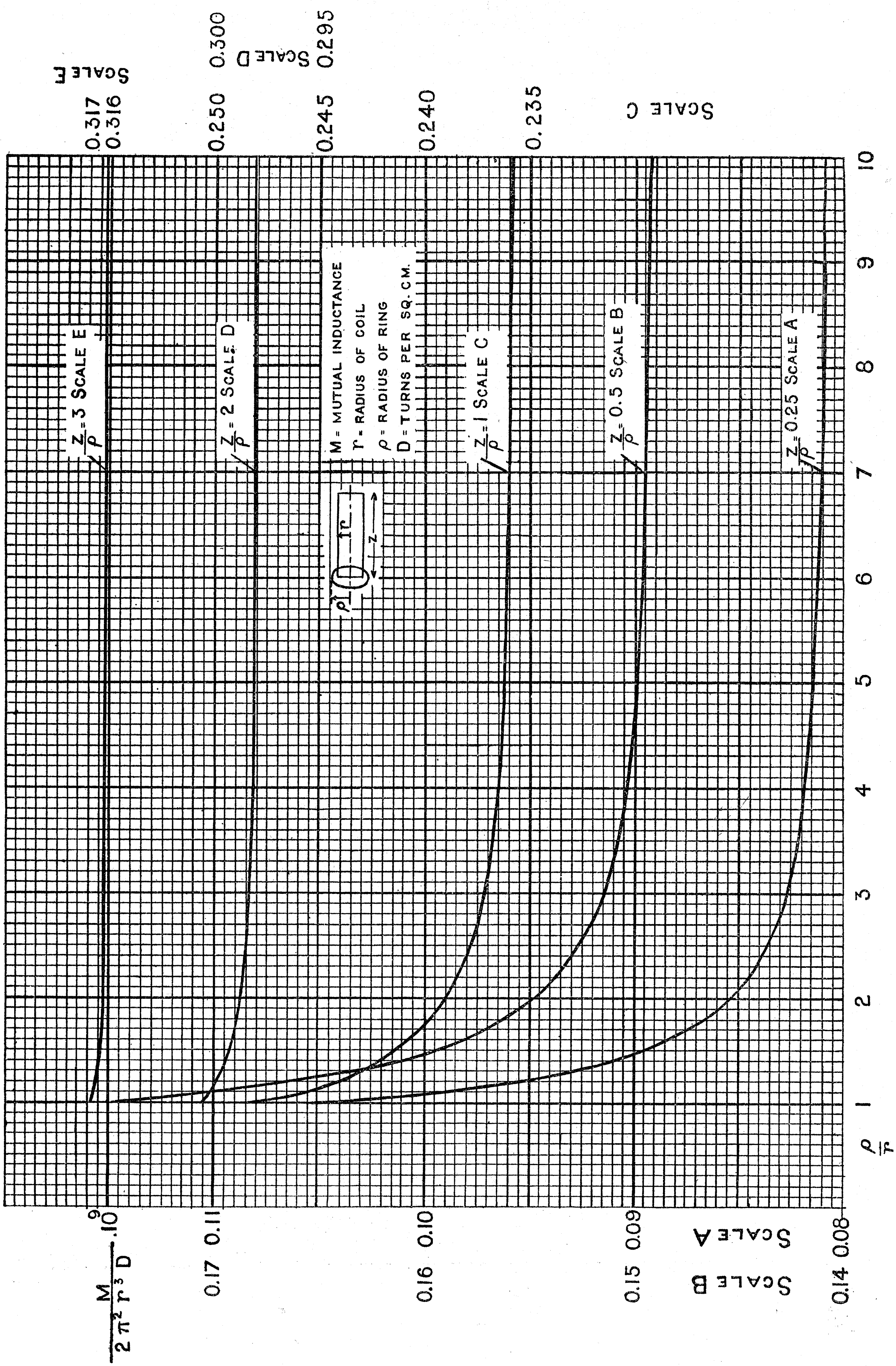

CHART IV.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

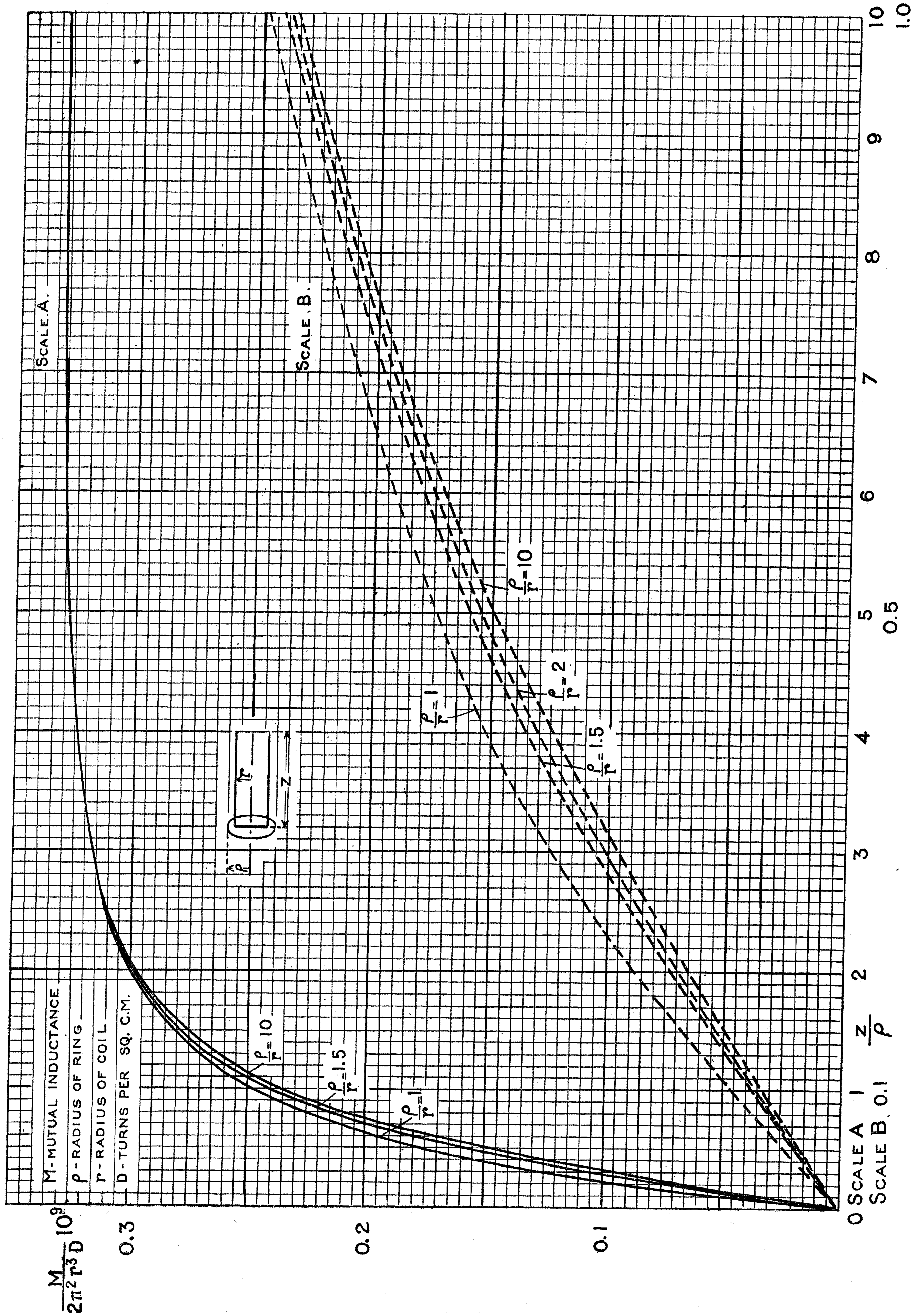
M - MUTUAL INDUCTANCE
ρ - RADIUS OF RING
r - RADIUS OF COIL
D - TURNS PER SQ. C.M.
SCALE A.
SCALE B.
ρ/r = 10
ρ/r = 1.5
ρ/r = 1
ρ/r = 2
M/(2π² r³ D) 10⁹
0.3
0.2
0.1
0 SCALE A 1 2 3 4 5 6 7 8 9 10
SCALE B 0.1 0.5 1.0
Z/ρ
ρ
r
Z

CHART V.

radius. The charts give the variation of M with ρ/r for six values of z/r up to 4 and the variation with z/r for ten values of ρ/r from 0·1 to 1·0. Charts IV and V give the mutual inductance of a solid coil and circle in the plane of the end face, the circle radius being larger than the coil radius. The charts give the variation of M for values of ρ/r from 1 to 10 and values of z/ρ from 0·25 to 10. The four charts should enable values of M to be obtained in general with an accuracy of at least one per cent.

(c) *The Application of the Charts to the Calculation of the Forces on Coils.*—We first consider a rectangular coil of internal radius a_1 and external radius a_2 and length $2b$. To determine completely the distribution of stress in the section of the coil, we need to know M at all points in the coil. If, now, we calculate M for a number of values of r/a_1 from 1 to a_2/a_1 on a number of planes A, B, C (fig. 1), then we can draw curves of constant M for the coil section.

For points in the plane A we subtract values of M due to a solid coil of radius a_1 and length $2b$ from the values for a solid coil of radius a_2 and length $2b$, or

$$\mathrm{M}(r, \mathrm{A}) = \mathrm{M}\left(\frac{r}{a_2}, \frac{2b}{a_2}\right) - \mathrm{M}'\left(\frac{r}{a_1}, \frac{2b}{a_1}\right),$$

M and M′ being obtained from the charts for which $\rho/r < 1$ and > 1 respectively. For the plane B we add the figures for coils of length $\frac{1}{2}b$ and $\frac{3}{2}b$ and radius a_2 and subtract the figures for the corresponding coils of radius a_1, so that

$$\mathrm{M}(r, \mathrm{B}) = \mathrm{M}\left(\frac{r}{a_2}, \frac{b}{2a_2}\right) + \mathrm{M}\left(\frac{r}{a_2}, \frac{3b}{2a_2}\right) - \mathrm{M}'\left(\frac{r}{a_1}, \frac{b}{2a_1}\right) - \mathrm{M}'\left(\frac{r}{a_1}, \frac{3b}{2a_1}\right).$$

Similarly for the plane C we have

$$\mathrm{M}(r, \mathrm{C}) = 2\mathrm{M}\left(\frac{r}{a_2}, \frac{b}{a_2}\right) - 2\mathrm{M}'\left(\frac{r}{a_1}, \frac{b}{a_1}\right).$$

For any other planes we proceed in a similar manner.

The resulting M curves, giving, of course, the lines of force of the coil, are given in fig. 8 for the first coil actually built.

From (16) we have for the stresses at any point

$$\left.\begin{aligned} \mathrm{P}_r &= \frac{\mathrm{I}^2\mathrm{D}}{2\pi r}\{\mathrm{M}(r, z) - \mathrm{M}(r_0, z)\} \\ \mathrm{P}_z &= \frac{\mathrm{I}^2\mathrm{D}}{2\pi r}\{\mathrm{M}(r, z) - \mathrm{M}(r, z_0)\} \end{aligned}\right\}, \quad \text{(H)}$$

I being the total current in the strip measured in absolute units. The first coil used had 9·25 turns per sq. cm. For a current of 30,000 amperes we have, therefore,

$$\mathrm{P} = 122\cdot5\,\frac{\Delta\,(\mathrm{M}/\mathrm{D})}{r}\ \text{kgms./sq. cm.}$$

Thus the radial compressive force at the point A whose radius is 2·8 cm. is 122·5 (130 — 11)/2·8 kgms./sq. cm. = 5200 kgms./sq. cm., or approximately 33 tons per sq. inch. Since P_z is here zero, the shearing force will be equal to P_r. At the point C we have

$$P_r = 8800 \text{ kgms./sq. cm.} \qquad P_z = 3800 \text{ kgms./sq. cm.},$$

so that the maximum shearing force $P_r - P_z = 5000$ kgms./sq. cm.

We see that along the inside surface of the coil M varies from 19 to 130, so that we are far from fulfilling our condition for minimum shearing force. We therefore proceed to determine the effect of a single step in the coil, the section chosen being illustrated in fig. 8*b*.

For points in the end plane A we take the M for the solid coil and subtract the M's for the section DENP, FGMN, HKLM, so that

$$\begin{aligned} M(r, A) = {} & M\left(\frac{r}{a_2}, \frac{2b}{a_2}\right) - M'\left(\frac{r}{a_3}, \frac{s}{a_3}\right) \\ & - \left\{M'\left(\frac{r}{a_3}, \frac{2b}{a_3}\right) - M'\left(\frac{r}{a_3}, \frac{2b-s}{a_3}\right)\right\} \\ & - \left\{M'\left(\frac{r}{a_1}, \frac{2b-s}{a_1}\right) - M'\left(\frac{r}{a_1}, \frac{s}{a_1}\right)\right\}. \end{aligned}$$

By a similar process of addition and subtraction we obtain the M's for points in the planes of B and C and so construct the M curves for the stepped coil.

These curves are given in fig. 8 and show clearly the effect of the step in reducing the maximum shearing stresses in the coil. For the point A the ΔM is now 60 as compared with 119 without the step, so that the shearing stress is halved and is now only 2600 kgms./sq. cm. The shearing stress for the point C corresponding to an M difference of 102 is now 4400 kgms./sq. cm., so that little reduction has been effected for this point. It is necessary, therefore, to make a further step in the coil to reduce the M values in the plane of the end. If the step is made by adding the sections RST, R′S′T′, then the effect on the M values for C and 0 is negligible and the value of M/D at R is now 88, giving an M difference for the point C of 69 and a shearing stress of 3000 kgms./sq. cm.

It is clear that in this way we can by trial and error very rapidly fix on a coil with a suitable section, bearing in mind, of course, the difficulties of winding a coil with too many steps in the section, and that these alterations mean a reduction in the efficiency of the coil.

4. *An Example of the Design of a Coil.*

From the preceding section it will be clear that the design of a coil is a compromise between different requirements. We are given a definite impressed electromotive force and circuit constants of the generator and have to produce a coil which will give the strongest possible field without a rise in temperature of its material greater than some definite figure. We have at the same time to select such a section for the coil as to reduce the shearing forces in the copper to a minimum. Unfortunately, a change in section giving a decrease in shearing forces means a reduction in the field attainable, so that these opposing factors have to be balanced one against the other, the final decision depending on one's judgment as to how far the material may be safely stressed, the breaking strength of the cadmium-copper alloy used being 7000 kgm./sq. cm. under tension. The process of design will, it is thought, be clarified if an actual example is given.

We take the following constants of the machine.

$$E = 2000,$$
$$\omega = 314,$$
$$R_2 = 2\cdot 53 \,.\, 10^{-2} \text{ ohms},$$
$$X_2 = 5\cdot 08 \,.\, 10^{-2}.$$

The fixed coil constants are

Internal radius $a_1 = 0\cdot 5$ cm.,
Temperature rise $\delta T = 160°$ C.,
Space factor $\lambda = 0\cdot 8$,
Mean resistivity $\rho = 2\cdot 4 \,.\, 10^{-6}$ ohms/cm³.

From E and G the first approximation to the volume is made

$$V = \frac{\pi}{\omega}\frac{E^2}{R^2} R_2 \,\zeta\left(\frac{X}{R}\right)\frac{1}{3\,.\,5\,.\,\delta T} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (32)$$

For the first approximation we take $R_1 = R_2$, $X_1 = X_2$, so that $X/R = 2$, and from fig. 3 $\zeta\,(2) = 0\cdot 20$ and $V = 141$ c.c.
From F, $V_1 = V/a_1^3\lambda = 1410$.

We now choose from chart I a point lying on the volume curve 1410, which has a maximum efficiency factor, and take $\alpha = 7\cdot 2$, $\beta = 4\cdot 2$. The temperature rise for this volume is now calculated, using the correct values of X_2 and R_2.

From (A)

$$\frac{X_2}{R_2} = \frac{\omega a_1^2 \lambda}{\rho} \,.\, 10^{-9}\, \phi_1.$$

From chart I $\phi_1 = 63$, so that $X_2/R_2 = 1\cdot 65$.

From (B)

$$R_2 = \frac{R_1^2 + X_1^2}{\sqrt{(1 + X_2^2/R_2^2)}} = 2\cdot 93 \,.\, 10^{-2}\,\Omega,$$
$$X_2 = 4\cdot 83 \,.\, 10^{-2}\,\Omega,$$

so that $X/R = 1\cdot 81$ and from fig. 3 $\zeta\,(X/R) = 0\cdot 22$.

Using E and G again the temperature rise for this coil is

$$\delta T = 175°.$$

The allowable temperature rise was 160° C., so that we now increase the volume in the ratio 175/160 and try a coil with a volume of 150 c.c. Using E we therefore select the volume curve 1500 on chart I and choose the point $\alpha = 7$, $\beta = 5$ as the point of greatest efficiency.

The temperature rise is now recalculated. Using A, since $\phi_1 = 65$, $X_2/R_2 = 1\cdot7$. From B, $R_2 = 2\cdot87 \,.\, 10^{-2}\,\Omega$, $X_2 = 4\cdot88 \,.\, 10^{-2}$ and $X/R = 1\cdot85$, $\zeta\,(X/R) = 0\cdot217$, and from E and G $\delta T = 162$.

The coil of external radius 3·5 cm. and axial length of 5·0 cm. therefore satisfies our requirements.

Maximum Field.—From D we find the value of the maximum field to be

$$H = \frac{E}{R}\sqrt{R_2}F\left(\frac{X}{R}\right)\sqrt{\left(\frac{\lambda}{a_1\rho}\right)}\,G_1.$$

From Fig. 3 F (1·84) = 0·614 and from chart I, $G_1 = 0\cdot152$, so that $H = 480{,}000$.

Wire Section.—From (2) we have $N = \sqrt{\left(\dfrac{4bc\lambda R_2}{2\pi\rho\,(a_1 + a_2)}\right)} = 107$.

Therefore section = 11·2 sq. mm.

Turns/sq. cm. = D = 7·13.

Stresses.—From H we have for the radial and axial stresses at the point C (fig. 1)

$$P_r = \frac{I^2D^2}{2\pi a_2}\left(\frac{M_C}{D} - \frac{M_0}{D}\right),$$

$$P_z = \frac{I^2D^2}{2\pi a_2}\left(\frac{M_C}{D} - \frac{M_A}{D}\right).$$

Using the charts as described we find

$$M_A/D = 235,$$
$$M_C/D = 352,$$
$$M_D/D = 26.$$

From C, $I = \dfrac{E}{R}\,F\left(\dfrac{X}{R}\right)$ and $F\left(\dfrac{X}{R}\right) = 0\cdot614$, so that

$$I = 2270 \text{ (absolute units)}.$$

Thus
$$P_r = 3980 \text{ kgms./sq. cm.},$$
$$P_z = 1430 \text{ kgms./sq. cm.},$$

and shearing force $= P_r - P_z = 2550$ kgms./sq. cm.

Alternative Coil with Lower Stresses.—To reduce the shearing stresses we obviously require a coil with a greater axial length and smaller radial depth, keeping approximately the same volume. From chart I, bearing in mind that points along the volume curve in the direction of increasing length have a smaller X/R and consequently greater heating, we select a point $\alpha = 6$, $\beta = 7\cdot2$, with a volume of 160 c.c.

For this coil we find $\delta T = 160° C.$

$$H = 458{,}000.$$

$$I = 22{,}400 \text{ amps.}$$

$$D = 7\cdot58.$$

$$P_r - P_z = 2150 \text{ kgms./sq. cm.}$$

The maximum shearing stress is thus reduced by 16 per cent. and the maximum field by 4·5 per cent.

The effect of making a step in the coil section has already been described and will not be worked out here.

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